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ON TWO APPROACHES TO THE STUDY OF THE EQUILIBRIUM STATES OF A NONHOLONOMIC MECHANICAL SYSTEM *

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Two approaches to the study of equilibrium of a nonholonomic system are compared. The first approach utilizes the Lagrange's equations with undertermined multipliers, and the second approach uses the Chaplygin or Voronets equations.

It is shown in /1/ that the application of the Chaplygin equations to his systems does not yield the equilibrium states of the second kind, and use of the Voronets equation does not always produce a solution to this problem. The present paper shows that using the equations (4) and (2), one can always choose a system of admissible vectors (9) which will enable the determination of all equilibrium states of the second kind. The difference in the results obtained is due to the fact that equations (6) determining the equilibrium states are not invariant under the change of the system of admissible vectors. Equations determining the manifold of the equilibrium states (14) invariant under the change of admissible vectors are obtained.

One of the approaches to the study of the position of equilibrium of a mechanical nonholonomic system acted upon by linear nonholonomic constraints, is based on the Lagrange's equations of motion with undetermined multipliers /1,2/

$$\frac{d}{dt}\frac{\partial T}{\partial q^{\cdot\varkappa}} - \frac{\partial T}{\partial q^{\varkappa}} = Q_\varkappa - \frac{\partial F}{\partial q^{\cdot\varkappa}} + \lambda_p\omega_\varkappa{}^p \tag{1}$$

$$\omega_\varkappa{}^p q^{\cdot\varkappa} = 0 \tag{2}$$

The indices $\lambda, \mu, \nu, \theta, \varkappa$ assume the values from 1 to n everywhere; p, q, r from $k+1$ to n, and a, b, c from 1 to k. The kinetic energy $T = {}^1/_2 g_{\lambda\mu} q^{\cdot\lambda} q^{\cdot\mu}$ is a quadratic form and the dissipative function $F(q, q^\cdot)$ is assumed to be a semi-definite quadratic form of the generalized velocities. The equilibrium states are obtained from the relations

$$Q_\varkappa(q, 0) + \lambda_p\omega_\varkappa{}^p(q) = 0 \tag{3}$$

Let us denote by $\alpha_a{}^\varkappa$ the coordinates of the admissible vectors /3/. We introduce the quantities $G_{ab} = g_{\lambda\mu}\alpha_a{}^\lambda\alpha_b{}^\mu, G^{pq} = g^{\lambda\mu}\omega_\lambda{}^p\omega_\mu{}^q$ and use them to obtain $\alpha_p{}^\varkappa = g^{\varkappa\nu}G_{pr}\omega_\nu{}^r$ and $\omega_\varkappa{}^a = g_{\varkappa\theta}G^{ab}\alpha_b{}^\theta$. It follows that the system (1) is equivalent to the following systems:

$$\left(\frac{d}{dt}\frac{\partial T}{\partial q^{\cdot\varkappa}} - \frac{\partial T}{\partial q^{\varkappa}} - Q_\varkappa + \frac{\partial F}{\partial q^{\cdot\varkappa}}\right)\alpha_a{}^\varkappa = 0 \tag{4}$$

$$\left(\frac{d}{dt}\frac{\partial T}{\partial q^{\cdot\varkappa}} - \frac{\partial T}{\partial q^{\varkappa}} - Q_\varkappa + \frac{\partial F}{\partial q^{\cdot\varkappa}}\right)\alpha_p{}^\varkappa = \lambda_p \tag{5}$$

Using the proposed method we can construct the characteristic equation without considering (5). Indeed, each equation of (5) has a corresponding column, in the determinant of the characteristic equation, in which one of the elements is unity and the rest are zero. Crossing out the columns and rows containing a unity leads to elimination of the corresponding equation.

Equations (4) form together with (2) a system of differential equations in terms of admissible vectors (3).

The second approach to the study of the equilibrium positions of a mechanical nonholonomic system involves use of the systems (2) and (4) /4,5/. The advantage of this approach consisits in a reduced number of equations. The conditions of equilibrium are written in this case in the form

$$Q_\varkappa\alpha_a{}^\varkappa = 0 \tag{6}$$

The Voronets equations are obtained from (4) and (2) for the following system of admissible vectors:

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$$\begin{array}{l}\alpha_1\,(1,\ 0,\ \ldots,\ 0,\ -a_1^{k+1},\ -a_1^{k+2},\ \ldots,\ -a_1^{n})\\ \alpha_2\,(0,\ 1,\ \ldots,\ 0,\ -a_2^{k+1},\ -a_2^{k+2},\ \ldots,\ -a_2^{n})\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \alpha_k\,(0,\ 0,\ \ldots,\ 1,\ -a_k^{k+1},\ -a_k^{k+2},\ \ldots,\ -a_k^{n})\end{array}$$

where the coefficients a_λ^p are given by the expressions

$$q^{\cdot p} = a_1^p q^{\cdot 1} + a_2^p q^{\cdot 2} + \ldots + a_k^p q^{\cdot k}$$

The Chaplygin equations represent a particular case of the Voronets equations when the expressions $T, Q_\varkappa = \partial u/\partial q_\varkappa, a_b^p, F$ are independent of the variables q^p. Every mechanical system has an infinite set of admissible vectors satisfying the condition $\omega_\varkappa^p \alpha_a^\varkappa = 0$. The passage from one system $\alpha_a^\varkappa$ to another $\alpha_{a'}^\varkappa$ is carried out with help of the formulas

$$\alpha_{a'}^\varkappa = \gamma_{a'}^a \alpha_a^\varkappa \quad (\det \| \gamma_{a'}^a \| \neq 0)$$

However, as was already recorded in /1/, use of the Chaplygin, and at times the Voronets equations in the study of equilibrium of a mechanical nonholonomic system does not lead to establishing the positions of equilibrium of the second kind of the system. We shall show how this can be taken into account when choosing the suitable admissible vectors.

Let us assume that $\omega_\varkappa^p$ are holomorphic functions of the variables $q^\varkappa$. We can always impart the same properties to $\alpha_a^\varkappa$. Taking into account the fact that (2) is independent, we can write their solutions in $q^{\cdot k+1}, \ldots, q^{\cdot n}$ as follows:

$$\varepsilon_a^p q^{\cdot a} + \delta_r^p \Delta q^{\cdot r} = 0 \tag{7}$$

$$\Delta = \det \| \omega_q^p \| \neq 0 \tag{8}$$

Condition (8) always holds when the indices of the generalized coordinates are suitably interchanged. The method of obtaining ε_a^p and Δ clearly implies that they are holomorphic functions of the generalized coordinates. We can choose

$$\alpha_a\,(\Delta\delta_a^1,\ \Delta\delta_a^2, \ldots, \Delta\delta_a^k,\ -\varepsilon_a^{k+1},\ -\varepsilon_a^{k+2}, \ldots, \varepsilon_a^{n}) \tag{9}$$

as the admissible vectors with holomorphic coordinates.

When the admissible vectors are chosen incorrectly, then, as was shown in /1/, the equilibrium state of the second kind may become inaccessible. Let us analyze the reasons of it. We write the equations (1) in the following form

$$q^{\cdot\cdot \rho} + \Gamma_{\lambda\mu}^\rho q^{\cdot\lambda} q^{\cdot\mu} = Q_\varkappa g^{\varkappa\rho} - \frac{\partial F}{\partial q^{\cdot\varkappa}} g^{\varkappa\rho} + \lambda_p \omega_\varkappa^p g^{\varkappa\rho} \tag{10}$$

and differentiate the constraint equations (2) with respect to time. Using (10) we obtain

$$\lambda_p G^{pq} = \Gamma_{\lambda\mu}^\rho \omega_\rho^q q^{\cdot\lambda} q^{\cdot\mu} - Q_\varkappa g^{\varkappa\rho} \omega_\rho^q - \frac{\partial \omega_\rho^q}{\partial q^\varkappa} q^{\cdot\varkappa} q^{\cdot\rho} + \frac{\partial F}{\partial q^{\cdot\varkappa}} g^{\varkappa\rho} \omega_\rho^q \tag{11}$$

Thus the Lagrange multipliers are given in the form of functions of the generalized coordinates $q^\varkappa$ and velocities $q^{\cdot\varkappa}$. In the state of equilibrium we have

$$\lambda_p = -G_{pq} Q_\varkappa g^{\varkappa\rho} \omega_\rho^q \tag{12}$$

Substituting (12) and (3) into (6) and using the relation

$$g^{\lambda\mu} = G^{ab} \alpha_a^\lambda \alpha_b^\mu + G^{pq} \alpha_p^\lambda \alpha_q^\mu \tag{13}$$

we obtain

$$Q_\varkappa \alpha_a^\varkappa \omega_\rho^a = 0 \tag{14}$$

The formula (14) yields the equations determining the manifolds of equilibrium states of the system. Every solution of (6) is also a solution of (14), although the converse is not always true. The change $\alpha_{a'} = \gamma_{a'}^a \alpha_a$ of the system of admissible vectors does not alter the conditions (14), but reduces the conditions (6) to the form

$$Q_\varkappa \alpha_a^\varkappa \gamma_{a'}^a = 0 \tag{15}$$

Conditions $Q_p = 0$ and $\alpha_a^b = \delta_a^b$ hold for the Chaplygin systems. In this case the condition (6) becomes $Q_a = 0$ and yields the equilibrium state of the first kind only /1/. Formula (14) yields all equilibrium states. The relations (6) are not invariant under the change of admissible vectors.

Formula (14) was derived under the condition that the equations of the system (2) are mutually independent and $\det \| g_{\lambda\mu} \| \neq 0$. We shall call the points at which the basic requirements are violated the singular points of the manifold of the equilibrium states.

We shall illustrate the above arguments by considering the problem of equilibrium of a Chaplygin sledge on a sloping plane /2/. We write the Lagrange's function

$$L = {}^1/_2 m[(x^{\cdot} + l\varphi^{\cdot}\cos\varphi)^2 + (y^{\cdot} + l\varphi^{\cdot}\sin\varphi)^2 + k^2\varphi^{\cdot 2}] - \mathrm{mg}\sin\alpha\,(y - l\cos\varphi)$$

and introduce the dissipative function

$$F = {}^1/_2 m[h(x^{\cdot 2} + y^{\cdot 2}) + h_1\varphi^{\cdot 2}]$$

The nonholonomic constraint is given by the equation $y^{\cdot} = x^{\cdot}\,\mathrm{tg}\,\varphi$.

We put $q^1 = \varphi,\ q^2 = x,\ q^3 = y$ and introduce the admissible vectors

$$\alpha_1\,(1, 0, 0),\ \alpha_2\,(0, \cos\varphi, \sin\varphi),\ \omega^3\,(0, -\sin\varphi, \cos\varphi) \tag{16}$$

The conditions of equilibrium are obtained, according to (6), in the form

$$\frac{\partial u}{\partial\varphi} = 0,\quad \frac{\partial u}{\partial x}\cos\varphi + \frac{\partial u}{\partial y}\sin\varphi = 0 \tag{17}$$

Using the relations

$$G^{11} = \frac{1}{mk^2},\quad G^{22} = \frac{l^2 + k^2}{mk^2},\quad G^{12} = G^{21} = -\frac{l}{mk^2}$$

we obtain from (14)

$$\frac{\partial u}{\partial\varphi} = 0\quad \left(\frac{\partial u}{\partial x}\cos\varphi + \frac{\partial u}{\partial y}\sin\varphi\right)\cos\varphi = 0,\quad \left(\frac{\partial u}{\partial x}\cos\varphi + \frac{\partial u}{\partial y}\sin\varphi\right)\sin\varphi = 0 \tag{18}$$

Solutions of (17) and (18) yield the same manifold of equilibrium states which was obtained in /2/. When the Chaplygin vectors

$$\alpha_1\,(1, 0, 0),\ \alpha_2\,(0, \mathrm{ctg}\,\varphi, 1) \tag{19}$$

are chosen according to the formula (6), we obtain

$$\partial u/\partial\varphi = 0, \partial u/\partial y = 0 \tag{20}$$

It appears that there are no values of the variables x, y and φ which satisfy the relations (20). Computing

$$G^{11} = \frac{1}{mk^2},\quad G^{22} = \frac{l^2 + k^2}{mk^2}\sin^2\varphi,\quad G^{12} = G^{21} = -\frac{l\sin\varphi}{mk^2}$$

and substituting them into (14), we again arrive at the equations (18). This is as expected, since we have established the invariance of (14). The passage from one set of admissible vectors to another, is determined by the system

$$\gamma_{1'}^1 = 1,\ \gamma_{1'}^2 = 0,\ \gamma_{2'}^1 = 0,\ \gamma_{2'}^2 = 1/\sin\varphi$$

The results obtained according to formula (6) using the admissible vectors (16) and (19) do not coincide, since the coefficient $\gamma_{2'}^2 = 1/\sin\varphi$, loses its meaning in the manifold of the equilibrium states.

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